Higher-order statistics for DSGE models

ONLINE APPENDIX

NOT FOR PUBLICATION

Contents

1. Exact expressions for pruned state-space representation

This is based on the technical appendix of Andreasen et al. (2014).

First we derive some additional expressions:

$$\hat{y}^f_{t+1} = g_x \hat{x}^f_t + g_u u_{t+1} \tag{1}$$

$$\hat{y}^s_{t+1} = g_x \hat{x}^s_t + \frac{1}{2}\left[G_{xx}\left(\hat{x}^f_t \otimes \hat{x}^f_t\right) + 2G_{xu}\left(\hat{x}^f_t \otimes u_{t+1}\right) + G_{uu}\left(u_{t+1} \otimes u_{t+1}\right) + g_{\sigma\sigma}\sigma^2\right] \tag{2}$$

$$\begin{aligned}\hat{y}^{rd}_{t+1} = {} & g_x \hat{x}^{rd}_t + G_{xx}\left(\hat{x}^f_t \otimes \hat{x}^s_t\right) + G_{xu}\left(\hat{x}^s_t \otimes u_{t+1}\right) + \frac{3}{6}G_{x\sigma\sigma}\hat{x}^f_t + \frac{3}{6}G_{u\sigma\sigma}u_{t+1} \\ & + \frac{1}{6}G_{xxx}\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes \hat{x}^f_t\right) + \frac{1}{6}G_{uuu}\left(u_{t+1} \otimes u_{t+1} \otimes u_{t+1}\right) \\ & + \frac{3}{6}G_{xxu}\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes u_{t+1}\right) + \frac{3}{6}G_{xuu}\left(\hat{x}^f_t \otimes u_{t+1} \otimes u_{t+1}\right)\end{aligned} \tag{3}$$

$$\hat{x}^f_{t+1} = h_x \hat{x}^f_t + h_u u_{t+1} \tag{4}$$

$$\hat{x}^s_{t+1} = h_x \hat{x}^s_t + \frac{1}{2}\left[H_{xx}\left(\hat{x}^f_t \otimes \hat{x}^f_t\right) + 2H_{xu}\left(\hat{x}^f_t \otimes u_{t+1}\right) + H_{uu}\left(u_{t+1} \otimes u_{t+1}\right) + h_{\sigma\sigma}\sigma^2\right] \tag{5}$$

$$\begin{aligned}\hat{x}^{rd}_{t+1} = {} & h_x \hat{x}^{rd}_t + H_{xx}\left(\hat{x}^f_t \otimes \hat{x}^s_t\right) + H_{xu}\left(\hat{x}^s_t \otimes u_{t+1}\right) + \frac{3}{6}H_{x\sigma\sigma}\hat{x}^f_t + \frac{3}{6}H_{u\sigma\sigma}u_{t+1} \\ & + \frac{1}{6}H_{xxx}\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes \hat{x}^f_t\right) + \frac{1}{6}H_{uuu}\left(u_{t+1} \otimes u_{t+1} \otimes u_{t+1}\right) \\ & + \frac{3}{6}H_{xxu}\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes u_{t+1}\right) + \frac{3}{6}H_{xuu}\left(\hat{x}^f_t \otimes u_{t+1} \otimes u_{t+1}\right)\end{aligned} \tag{6}$$

$$\begin{aligned}\left(\hat{x}^f_{t+1} \otimes \hat{x}^f_{t+1}\right) = {} & \left(h_x \otimes h_x\right)\left(\hat{x}^f_t \otimes \hat{x}^f_t\right) + \left(h_u \otimes h_u\right)\left(u_{t+1} \otimes u_{t+1} - \Gamma_{2u} + \Gamma_{2u}\right) \\ & + \left(h_x \otimes h_u\right)\left(\hat{x}^f_t \otimes u_{t+1}\right) + \left(h_u \otimes h_x\right)\left(u_{t+1} \otimes \hat{x}^f_t\right)\end{aligned} \tag{7}$$

$$\begin{aligned}\left(\hat{x}^f_{t+1} \otimes \hat{x}^s_{t+1}\right) = {} & \left(h_x \otimes \frac{\sigma^2}{2}h_{\sigma\sigma}\right)\hat{x}^f_t + \left(h_u \otimes \frac{\sigma^2}{2}h_{\sigma\sigma}\right)u_{t+1} \\ & + \left(h_x \otimes h_x\right)\left(\hat{x}^f_t \otimes \hat{x}^s_t\right) + \left(h_u \otimes h_x\right)\left(u_{t+1} \otimes \hat{x}^s_t\right) \\ & + \left(h_x \otimes \frac{1}{2}H_{xx}\right)\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes \hat{x}^f_t\right) + \left(h_u \otimes \frac{1}{2}H_{uu}\right)\left(u_{t+1} \otimes u_{t+1} \otimes u_{t+1} - \Gamma_{3u} + \Gamma_{3u}\right) \\ & + \left(h_x \otimes \frac{1}{2}H_{uu}\right)\left(\hat{x}^f_t \otimes u_{t+1} \otimes u_{t+1}\right) + \left(h_u \otimes \frac{1}{2}H_{xu}\right)\left(u_{t+1} \otimes \hat{x}^f_t \otimes u_{t+1}\right) \\ & + \left(h_x \otimes H_{xu}\right)\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes u_{t+1}\right) + \left(h_u \otimes H_{xx}\right)\left(u_{t+1} \otimes \hat{x}^f_t \otimes \hat{x}^f_t\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\left(\hat{x}^f_{t+1} \otimes \hat{x}^f_{t+1} \otimes \hat{x}^f_{t+1}\right) = {} & \left(h_x \otimes h_x \otimes h_x\right)\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes \hat{x}^f_t\right) + \left(h_x \otimes h_u \otimes h_u\right)\left(\hat{x}^f_t \otimes u_{t+1} \otimes u_{t+1}\right) \\ & + \left(h_x \otimes h_x \otimes h_u\right)\left(\hat{x}^f_t \otimes \hat{x}^f_t \otimes u_{t+1}\right) + \left(h_x \otimes h_u \otimes h_x\right)\left(\hat{x}^f_t \otimes u_{t+1} \otimes \hat{x}^f_t\right) \\ & + \left(h_u \otimes h_x \otimes h_x\right)\left(u_{t+1} \otimes \hat{x}^f_t \otimes \hat{x}^f_t\right) + \left(h_u \otimes h_u \otimes h_u\right)\left(u_{t+1} \otimes u_{t+1} \otimes u_{t+1} - \Gamma_{3u} + \Gamma_{3u}\right) \\ & + \left(h_u \otimes h_x \otimes h_u\right)\left(u_{t+1} \otimes \hat{x}^f_t \otimes u_{t+1}\right) + \left(h_u \otimes h_u \otimes h_x\right)\left(u_{t+1} \otimes u_{t+1} \otimes \hat{x}^f_t\right)\end{aligned} \tag{9}$$

1.1. State-space system of first-order approximation

In a first-order approximation the system dynamics are captured by equations (1) and (4), we are therefore already working in a linear state-space system. That is, define $z_t := \hat{x}^f_t$, $y_t := \hat{y}^f_t + \bar{y}$, $\xi_{t+1} := u_{t+1}$, $c := 0$, $d := 0$, $A := h_x$, $B := h_u$, $C := g_x$ and $D := g_u$, then the equations can be rewritten as

$$\begin{aligned} z_{t+1} &= c + Az_t + B\xi_{t+1} \\ y_{t+1} &= \bar{y} + d + Cz_t + D\xi_{t+1} \end{aligned}$$

Note that if u_t is Gaussian, ξ_t is clearly Gaussian as well.

1.2. State-space system of second-order approximation and pruning

In a second-order approximation the system dynamics are captured by equations (1), (2), (4), (5) and (7). To set up the pruned state-space system we define

$$y_t = \hat{y}_t^f + \hat{y}_t^s + \bar{y}, \qquad z_t := \begin{pmatrix} \hat{x}_t^f \\ \hat{x}_t^s \\ \hat{x}_t^f \otimes \hat{x}_t^f \end{pmatrix}, \qquad \xi_{t+1} := \begin{pmatrix} u_{t+1} \\ u_{t+1} \otimes u_{t+1} - \Gamma_{2u} \\ \hat{x}_t^f \otimes u_{t+1} \\ u_{t+1} \otimes \hat{x}_t^f \end{pmatrix}$$

and

$$A := \begin{pmatrix} h_x & 0 & 0 \\ 0 & h_x & \frac{1}{2} H_{xx} \\ 0 & 0 & h_x \otimes h_x \end{pmatrix}, \qquad B := \begin{pmatrix} h_u & 0 & 0 & 0 \\ 0 & \frac{1}{2} H_{uu} & H_{xu} & 0 \\ 0 & h_u \otimes h_u & h_x \otimes h_u & h_u \otimes h_x \end{pmatrix},$$

$$C := \begin{pmatrix} g_x & g_x & \frac{1}{2} G_{xx} \end{pmatrix} \qquad D := \begin{pmatrix} g_u & \frac{1}{2} G_{uu} & G_{xu} & 0 \end{pmatrix}$$

$$c := \begin{pmatrix} 0 \\ \frac{1}{2}(h_{\sigma\sigma}\sigma^2 + H_{uu}\Gamma_{2,u} \\ (h_u \otimes h_u)\Gamma_{2,u} \end{pmatrix} \qquad d := \begin{pmatrix} \frac{1}{2} g_{\sigma\sigma}\sigma^2 + \frac{1}{2} G_{uu}\Gamma_{2,u} \end{pmatrix}$$

The system can thus be rewritten as a linear state-space representation

$$\begin{aligned} z_{t+1} &= c + A z_t + B \xi_{t+1} \\ y_{t+1} &= \bar{y} + d + C z_t + D \xi_{t+1} \end{aligned}$$

Note that even if u_t is Gaussian, ξ_t is clearly non-Gaussian.

1.3. State-space system of third-order approximation and pruning

In a third-order approximation the system dynamics are captured by equations (1), (2), (3), (4), (5),(6), (7), (8) and (9). To set up the pruned state-space system we define

$$y_t = \hat{y}_t^f + \hat{y}_t^s + \hat{y}_t^{rd} + \bar{y}, \qquad z_t := \begin{pmatrix} \hat{x}_t^f \\ \hat{x}_t^s \\ \hat{x}_t^f \otimes \hat{x}_t^f \\ \hat{x}_t^{rd} \\ \hat{x}_t^f \otimes \hat{x}_t^s \\ \hat{x}_t^f \otimes \hat{x}_t^f \otimes x_t^f \end{pmatrix}, \qquad \xi_{t+1} := \begin{pmatrix} u_{t+1} \\ u_{t+1} \otimes u_{t+1} - \Gamma_{2u} \\ \hat{x}_t^f \otimes u_{t+1} \\ u_{t+1} \otimes \hat{x}_t^f \\ \hat{x}_t^s \otimes u_{t+1} \\ u_{t+1} \otimes \hat{x}_t^s \\ \hat{x}_t^f \otimes \hat{x}_t^f \otimes u_{t+1} \\ \hat{x}_t^f \otimes u_{t+1} \otimes \hat{x}_t^f \\ u_{t+1} \otimes \hat{x}_t^f \otimes \hat{x}_t^f \\ \hat{x}_t^f \otimes u_{t+1} \otimes u_{t+1} \\ u_{t+1} \otimes \hat{x}_t^f \otimes u_{t+1} \\ u_{t+1} \otimes u_{t+1} \otimes \hat{x}_t^f \\ u_{t+1} \otimes u_{t+1} \otimes u_{t+1} - \Gamma_{3,u} \end{pmatrix}$$

and

$$A := \begin{pmatrix} h_x & 0 & 0 & 0 & 0 & 0 \\ 0 & h_x & \frac{1}{2} H_{xx} & 0 & 0 & 0 \\ 0 & 0 & h_x \otimes h_x & 0 & 0 & 0 \\ \frac{3}{6} H_{x\sigma\sigma}\sigma^2 & 0 & 0 & h_x & H_{xx} & \frac{1}{6} H_{xxx} \\ h_x \otimes \frac{1}{2} h_{\sigma\sigma}\sigma^2 & 0 & 0 & 0 & h_x \otimes h_x & h_x \otimes \frac{1}{2} H_{xx}) \\ 0 & 0 & 0 & 0 & 0 & h_x \otimes h_x \otimes h_x)]; \end{pmatrix}$$

$$
B := \begin{pmatrix}
h_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}H_{uu} & H_{xu} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h_u \otimes h_u & h_x \otimes h_u & h_u \otimes h_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{6}H_{u\sigma\sigma}\sigma^2 & 0 & 0 & 0 & H_{xu} & 0 & \frac{3}{6}H_{xxu} & 0 & 0 & \frac{3}{6}H_{xuu} & 0 & 0 & \frac{1}{6}H_{uuu} \\
h_u \otimes \frac{1}{2}h_{\sigma\sigma}\sigma^2 & 0 & 0 & 0 & 0 & h_u \otimes h_x & h_x \otimes H_{xu} & 0 & h_u \otimes \frac{1}{2}H_{xx} & h_x \otimes \frac{1}{2}H_{uu} & h_u \otimes H_{xu} & 0 & h_u \otimes \frac{1}{2}H_{uu} \\
0 & 0 & 0 & 0 & 0 & 0 & h_x \otimes h_x \otimes h_u & h_x \otimes h_u \otimes h_x & h_u \otimes h_x \otimes h_x & h_x \otimes h_u \otimes h_u & h_u \otimes h_x \otimes h_u & h_u \otimes h_u \otimes h_x & h_u \otimes h_u \otimes h_u
\end{pmatrix},
$$

$$
\begin{aligned}
C &:= \begin{pmatrix} g_x + \frac{1}{2}G_{x\sigma\sigma}\sigma^2 & g_x & \frac{1}{2}G_{xx} & g_x & G_{xx} & \frac{1}{6}G_{xxx} \end{pmatrix} \\
D &:= \begin{pmatrix} g_u + \frac{1}{2}G_{u\sigma\sigma}\sigma^2 & \frac{1}{2}G_{uu} & G_{xu} & 0 & G_{xu} & 0 & \frac{1}{2}G_{xxu} & 0 & 0 & \frac{1}{2}G_{xuu} & 0 & 0 & \frac{1}{6}G_{uuu} \end{pmatrix} \\
c &:= \begin{pmatrix} 0 \\ \frac{1}{2}h_{\sigma\sigma}\sigma^2 + \frac{1}{2}H_{uu}\Gamma_{2,u} \\ (h_u \otimes h_u)\Gamma_{2,u} \\ \frac{1}{6}H_{uuu}\Gamma_{3,u} + \frac{1}{6}H_{\sigma\sigma\sigma}\sigma^3 \\ (h_u \otimes \frac{1}{2}H_{uu})\Gamma_{3,u} \\ (h_u \otimes h_u \otimes h_u)\Gamma_{3,u} \end{pmatrix} \\
d &:= \begin{pmatrix} \frac{1}{2}g_{\sigma\sigma}\sigma^2 + \frac{1}{2}G_{uu}\Gamma_{2,u} + \frac{1}{6}G_{uuu}\Gamma_{3,u} + \frac{1}{6}G_{\sigma\sigma\sigma}\sigma^3 \end{pmatrix}
\end{aligned}
$$

The system can thus be rewritten as a linear state-space representation

$$
\begin{aligned}
z_{t+1} &= c + Az_t + B\xi_{t+1} \\
y_{t+1} &= \bar{y} + d + Cz_t + D\xi_{t+1}
\end{aligned}
$$

Note that even if u_t is Gaussian, ξ_t is clearly non-Gaussian, since it's higher-order cumulants are nonzero.

2. Computation of product moments for extended innovations

2.1. First-order approximation

Given a first-order approximation, the innovations are defined as the $n_\xi \times 1$ vector $\xi_{t+1} = u_{t+1}$ with $n_\xi = n_u$ elements. We are interested in product moments $M_{2,\xi} := E(\xi_t \otimes \xi_t)$, $M_{3,\xi} := E(\xi_t \otimes \xi_t \otimes \xi_t)$ and $M_{4,\xi} := E(\xi_t \otimes \xi_t \otimes \xi_t \otimes \xi_t)$ with n_ξ^2, n_ξ^3 and n_ξ^4 elements, respectively. These, however, contain many duplicate elements. Denote with $\widetilde{M}_{k,\xi}$ the unique elements of $M_{k,\xi}$, we have the following relationships:

$$
M_{2,\xi} = DP_{n_\xi} \cdot \widetilde{M}_{2,\xi}, \quad M_{3,\xi} = TP_{n_\xi} \cdot \widetilde{M}_{3,\xi}, \quad M_{4,\xi} = QP_{n_\xi} \cdot \widetilde{M}_{4,\xi},
$$

with the duplication matrix DP_{n_ξ} defined by Magnus & Neudecker (1999), and the triplication matrix TP_{n_ξ} and quadruplication matrix QP_{n_ξ} similarly defined by Meijer (2005). Note that these matrices are independent of θ and their Moore-Penrose-Inverse always exists, e.g. $(QP_{n_\xi}' QP_{n_\xi})^{-1} QP_{n_\xi}' \cdot M_{4,\xi} = \widetilde{M}_{4,\xi}$. Further, DP_{n_ξ}, TP_{n_ξ} and QP_{n_ξ} are constructed such that there is a unique ordering in $\widetilde{M}_{k,\xi}$, see Meijer (2005) for an example and more details.

To compute the product-moments of ξ_t symbolically we therefore use the following procedure in Matlab given the number of shocks n_u and the order of product moments k=2,3,4.

1. Define $u_{t+1} = (u_{t+1,1}, \dots u_{t+1,n_u})'$ and $\Sigma_u = [sig_{ij}]_{nu \times nu}$ symbolically with $i, j = 1, \dots n_u$.
2. Get all integer permutations of $[i_1, i_2, \dots i_{n_\xi}]$ that sum up to k, with $i_j = 1, \dots, k$ and $j = 1, \dots, n_\xi$. Sort them in the ordering of Meijer (2005).
3. For each permutation $[i_1, i_2, \dots i_{n_\xi}]$ evaluate symbolically

$$
E\left[(\xi_{1,t})^{i_1} \cdot (\xi_{2,t})^{i_2} \cdot \ldots (\xi_{n_\xi,t})^{i_{n_\xi}} \right]
$$

and store it in the vector $\widetilde{M}_{k,\xi}$.

The expressions we get in step 3 contain terms of the form

$$const. \cdot E[(u_{1,t+1})^{i_{u_1}} \cdot (u_{2,t+1})^{i_{u_2}} \cdot \cdot \cdot (u_{n_u,t+1})^{i_{u_{n_u}}}],$$

that is joint product moments of the elements of u_{t+1}. Given a function that evaluates the moment structure of u_{t+1} either analytically or numerically, we are able to calculate these terms individually and save them into script files. Note, that these computations need only to be done once for a model, after that we simply evaluate the script files numerically given model parameters θ. Our code can evaluate product moments from the Gaussian as well as Student-t distribution.

Normal distribution. In the case that u_t is normally distributed, the joint product moments are functions of the variances and covariances in Σ and can be computed analytically. To this end, we use the very efficient method and Matlab function of Kan (2008) to derive these joint product moments symbolically. The cumulants can then be computed as outlined in the paper.

Student's t distribution. In the case that u_t is Student-t distributed with v degrees of freedom, we rewrite u_t in terms of a Inverse-Gamma distributed variable $W = v^{-1/2} \sim IGAM(v/2, v/2)$, and a normally distributed variable $\varepsilon_t \sim N(0, \Sigma)$, $u_t = v^{-1/2}\varepsilon_t$ (similar to Kotz & Nadarajah (2004) or Roth (2013)). Since W and ε_t are independent, we have $E(u_t u_t') = E(W)E(\varepsilon_t\varepsilon_t') = \frac{v}{v-2}\Sigma$. Whereas all odd product moments of u_t are zero, the even product moments ($n = \sum_{j=1}^{n_u} i_{u_j}$ is an even number) are given by

$$E[(u_{1,t})^{i_{u_1}} \cdot (u_{2,t})^{i_{u_2}} \cdot \cdot \cdot (u_{n_u,t})^{i_{u_{n_u}}}] = E[W^{\frac{n}{2}}] \cdot E[(\varepsilon_{1,t})^{i_{u_1}} \cdot (\varepsilon_{2,t})^{i_{u_2}} \cdot \cdot \cdot (\varepsilon_{n_u,t})^{i_{u_{n_u}}}].$$

The first term is equal to $E[W^k] = \frac{(v/2)^k}{(v/2-1)\ldots(v/2-k)}$ and since ε_t is multivariate normal, we can use Kan (2008)'s procedure and Matlab function for the second product. The cumulants can then be computed as outlined in the paper.

2.2. Second-order approximation

Given a second-order approximation, the innovations are defined as the $n_\xi \times 1$ vector

$$\xi_{t+1} = \begin{pmatrix} u_{t+1}' & (u_{t+1} \otimes u_{t+1} - vec(\Sigma))' & (x_t^f \otimes u_{t+1})' & (u_{t+1} \otimes x_t^f)' \end{pmatrix}'$$

with $n_\xi = n_u + n_u^2 + 2n_x n_u$ elements. We are interested in product moments $M_{2,\xi} := E(\xi_t \otimes \xi_t)$, $M_{3,\xi} := E(\xi_t \otimes \xi_t \otimes \xi_t)$ and $M_{4,\xi} := E(\xi_t \otimes \xi_t \otimes \xi_t \otimes \xi_t)$ with n_ξ^2, n_ξ^3 and n_ξ^4 elements, respectively. In order to compute these objects efficiently, we first reduce the dimension of ξ_t, since it has some duplicate elements. That is, we compute product-moments for the $n_{\tilde{\xi}} = n_u + n_u(n_u+1)/2 + n_u n_x$ vector

$$\tilde{\xi}_{t+1} := \begin{pmatrix} u_{t+1}' & (DP_{n_u}^+(u_{t+1} \otimes u_{t+1} - vec(\Sigma)))' & (x_t^f \otimes u_{t+1})' \end{pmatrix}'$$

since

$$\xi_t = \begin{pmatrix} I & 0 & 0 \\ 0 & DP_{n_u} & 0 \\ 0 & 0 & I \\ 0 & 0 & K_{n_u,n_x} \end{pmatrix} \tilde{\xi}_t := F_\xi \cdot \tilde{\xi}_t$$

with $DP_{n_u}^+$ being the Moore-Penrose-Inverse of the duplication matrix DP_{n_u} and K_{n_u,n_x} the commutation matrix such that $K_{n_u,n_x}(x_t^f \otimes u_{t+1}) = (u_{t+1} \otimes x_t^f)$. Then we have

$$M_{k,\xi} := [\otimes_{j=1}^k F_\xi] \cdot M_{k,\tilde{\xi}}$$

denoting the k-th (k=2,3,4)-order product moment of $\tilde{\xi}_t$. Since $[\otimes_{j=1}^{k} F_{\xi}]$ does not change with θ, we can focus on $M_{k,\tilde{\xi}}$. $M_{k,\tilde{\xi}}$, however, contains also many duplicate elements. Denote with $\widetilde{M}_{k,\tilde{\xi}}$ the unique elements of $M_{k,\tilde{\xi}}$, we have the following relationships:

$$M_{2,\tilde{\xi}} = DP_{n_{\tilde{\xi}}} \cdot \widetilde{M}_{2,\tilde{\xi}}, \quad M_{3,\tilde{\xi}} = TP_{n_{\tilde{\xi}}} \cdot \widetilde{M}_{3,\tilde{\xi}}, \quad M_{4,\tilde{\xi}} = QP_{n_{\tilde{\xi}}} \cdot \widetilde{M}_{4,\tilde{\xi}},$$

with the duplication matrix $DP_{n_{\tilde{\xi}}}$ defined by Magnus & Neudecker (1999), and the triplication matrix $TP_{n_{\tilde{\xi}}}$ and quadruplication matrix $QP_{n_{\tilde{\xi}}}$ similarly defined by Meijer (2005).[1] Note that these matrices are independent of θ and their Moore-Penrose-Inverse always exists, e.g. $(QP'_{n_{\tilde{\xi}}} QP_{n_{\tilde{\xi}}})^{-1} QP'_{n_{\tilde{\xi}}} \cdot M_{4,\tilde{\xi}} = \widetilde{M}_{4,\tilde{\xi}}$. Further, $DP_{n_{\tilde{\xi}}}$, $TP_{n_{\tilde{\xi}}}$ and $QP_{n_{\tilde{\xi}}}$ are constructed such that there is a unique ordering in $\widetilde{M}_{k,\tilde{\xi}}$, see Meijer (2005) for an example and more details.

To compute the product-moments of $\tilde{\xi}_t$ symbolically we therefore use the following procedure in Matlab given the number of shocks n_u, the number of state variables n_x and the order of product moments k=2,3,4.

1. Define $u_{t+1} = (u_{t+1,1}, \dots u_{t+1,n_u})'$, $x_t^f = (x_{t,1}^f, \dots x_{t,n_x}^f)'$ and $\Sigma_u = [sig_{ij}]_{nu \times nu}$ symbolically with $i, j = 1, \dots n_u$. Set up
$$\tilde{\xi}_t = (u_t', DP_{n_u}^{+}(u_{t+1} \otimes u_{t+1} - vec(\Sigma))', (x_t^f \otimes u_{t+1})')'.$$
2. Get all integer permutations of $[i_1, i_2, \dots i_{n_{\tilde{\xi}}}]$ that sum up to k, with $i_j = 1, \dots, k$ and $j = 1, \dots, n_{\tilde{\xi}}$. Sort them in the ordering of Meijer (2005).
3. For each permutation $[i_1, i_2, \dots i_{n_{\tilde{\xi}}}]$ evaluate symbolically
$$E\left[(\tilde{\xi}_{1,t})^{i_1} \cdot (\tilde{\xi}_{2,t})^{i_2} \cdot \ldots (\tilde{\xi}_{n_{\tilde{\xi}},t})^{i_{n_{\tilde{\xi}}}}\right]$$
and store it in the vector $\widetilde{M}_{k,\xi}$.
4. Optionally: Use Matlab's `unique` function to further reduce the dimension of $\widetilde{M}_{k,\xi}$.

The expressions we get in step 3 contain terms of the form

$$const. \cdot E[(u_{1,t+1})^{i_{u_1}} \cdot (u_{2,t+1})^{i_{u_2}} \cdot \cdot (u_{n_u,t+1})^{i_{u_{n_u}}}] \cdot E[(x_{1,t}^f)^{i_{x_1}} \cdot (x_{2,t}^f)^{i_{x_2}} \cdot \cdot (x_{n_x,t}^f)^{i_x^{n_x}}],$$

that is joint product moments of the elements of u_{t+1} and x_t^f (keeping in mind that x_t^f and u_{t+1} are independent due to the temporal independence of u_t). For instance, for $n_u = n_x = 1$ the third-order product moment of $\tilde{\xi}_t$ is equal to

$$\tilde{M}_{3,\xi} = vec\left(E\begin{bmatrix} u^3 & u^4 - \sigma_u^2 u^2 \\ u^3 x & \sigma_u^4 u - 2\sigma_u^2 u^3 + u^5 \\ xu^4 - \sigma_u^2 x u^2 & u^3 x^2 \\ -\sigma_u^6 + 3\sigma_u^4 u^2 - 3\sigma_u^2 u^4 + u^6 & x\sigma_u^4 u - 2x\sigma_u^2 u^3 + xu^5 \\ u^4 x^2 - \sigma_u^2 u^2 x^2 & u^3 x^3 \end{bmatrix}'\right)$$

where we dropped sub- and superscripts and $E(u^2) = \sigma_u^2$. Given a function that evaluates the moment structure of x_t^f and u_{t+1} either analytically or numerically, we are able to calculate these terms individually and save them into script files. Note, that these computations need only to be done once for a model, after that we simply evaluate the script files numerically given model parameters θ. Our code can evaluate product moments from the Gaussian as well as Student-t distribution.

[1] Actually $\widetilde{M}_{k,\tilde{\xi}}$ has some further duplicate terms for $n_u, n_x > 1$ due to higher-order cross terms of u_{t+1} and x_t^f, which we can further reduce using indices from the `unique` function of Matlab.

Normal distribution. In the case that u_t is normally distributed, x_t^f is also Gaussian with covariance matrix Σ_x. Therefore,

$$\begin{pmatrix} u_{t+1} \\ x_t^f \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma_x \end{pmatrix}\right)$$

is multivariate normal. All joint product moments are therefore functions of the variances and covariances in Σ and Σ_x and can be computed analytically. To this end, we use the very efficient method and Matlab function of Kan (2008) to derive these joint product moments symbolically. For our example with $n_u = n_x = 1$ and Gaussian u_t, we get the unique entries

$$\begin{aligned}
\widetilde{M}_{2,\xi} &= \left[\sigma_u^2,\ 0,\ 0,\ 2\sigma_u^4,\ 0,\ \sigma_u^2\sigma_x^2\right]' \\
\widetilde{M}_{3,\xi} &= \left[0,\ 2\sigma_u^4,\ 0,\ 0,\ 0,\ 0,\ 8\sigma_u^6,\ 0,\ 2\sigma_u^4\sigma_x^2,\ 0\right]' \\
\widetilde{M}_{4,\xi} &= \left[3\sigma_u^4,\ 0,\ 0,\ 10\sigma_u^6,\ 0,\ 3\sigma_u^4\sigma_x^2,\ 0,\ 0,\ 0,\ 0,\ 60\sigma_u^8,\ 0,\ 10\sigma_u^6\sigma_x^2,\ 0,\ 9\sigma_u^4\sigma_x^4\right]'
\end{aligned}$$

where $E(x_t^{f2}) = \sigma_x^2$. The cumulants can then be computed as outlined in the paper. Since the third-order cumulant of a Gaussian process must be zero, we now see, that ξ_t is clearly non-Gaussian, since its third-order cumulant is different from zero, even if the underlying distribution for u_t is Gaussian.

Student's t distribution. In the case that u_t is Student-t distributed with v degrees of freedom, we rewrite u_t in terms of a Inverse-Gamma distributed variable $W = v^{-1/2} \sim IGAM(v/2, v/2)$, and a normally distributed variable $\varepsilon_t \sim N(0, \Sigma)$, $u_t = v^{-1/2}\varepsilon_t$ (similar to Kotz & Nadarajah (2004) or Roth (2013)). Since W and ε_t are independent, we have $E(u_t u_t') = E(W)E(\varepsilon_t\varepsilon_t') = \frac{v}{v-2}\Sigma$. Whereas all odd product moments of u_t are zero, the even product moments ($n = \sum_{j=1}^{n_u} i_{u_j}$ is an even number) are given by

$$E[(u_{1,t})^{i_{u_1}} \cdot (u_{2,t})^{i_{u_2}} \cdot \cdot \cdot (u_{n_u,t})^{i_{u_{n_u}}}] = E[W^{\frac{n}{2}}] \cdot E[(\varepsilon_{1,t})^{i_{u_1}} \cdot (\varepsilon_{2,t})^{i_{u_2}} \cdot \cdot \cdot (\varepsilon_{n_u,t})^{i_{u_{n_u}}}].$$

The first term is equal to $E[W^k] = \frac{(v/2)^k}{(v/2-1)\dots(v/2-k)}$ and since ε_t is multivariate normal, we can use Kan (2008)'s procedure and Matlab function for the second product. Similar arguments apply to the product moments of x_t^f, for instance the variance is given by

$$vec(\Sigma_x) = E[x_t^f \otimes x_t^f] = \underbrace{E[W]}_{\frac{v}{v-2}} \cdot (I_{n_x^2} - h_x \otimes h_x)^{-1}(h_u \otimes h_u) \cdot \underbrace{E[\varepsilon_t \otimes \varepsilon_t]}_{vec(\Sigma)}.$$

Thus, odd product moments are also zero, whereas even product moments can also be computed symbolically by Kan (2008)'s procedure and Matlab function, however, adjusted for $E[W^{n/2}]$. The cumulants can then be computed as outlined in the paper.

2.3. Third-order approximation

Given a third-order approximation, the innovations are defined as the $n_\xi \times 1$ vector

$$\xi_{t+1} := \begin{pmatrix}
u_{t+1} \\
u_{t+1} \otimes u_{t+1} - \Gamma_{2u} \\
\hat{x}_t^f \otimes u_{t+1} \\
u_{t+1} \otimes \hat{x}_t^f \\
\hat{x}_t^s \otimes u_{t+1} \\
u_{t+1} \otimes \hat{x}_t^s \\
\hat{x}_t^f \otimes \hat{x}_t^f \otimes u_{t+1} \\
\hat{x}_t^f \otimes u_{t+1} \otimes \hat{x}_t^f \\
u_{t+1} \otimes \hat{x}_t^f \otimes \hat{x}_t^f \\
\hat{x}_t^f \otimes u_{t+1} \otimes u_{t+1} \\
u_{t+1} \otimes \hat{x}_t^f \otimes u_{t+1} \\
u_{t+1} \otimes u_{t+1} \otimes \hat{x}_t^f \\
u_{t+1} \otimes u_{t+1} \otimes u_{t+1} - \Gamma_{3,u}
\end{pmatrix}$$

with $n_\xi = n_u + n_u^2 + 2n_x n_u + 2n_x n_u + 3n_x^2 n_u + 3n_x n_u^2 + n_u^2$ elements. We are interested in product moments $M_{2,\xi} := E(\xi_t \otimes \xi_t)$, $M_{3,\xi} := E(\xi_t \otimes \xi_t \otimes \xi_t)$ and $M_{4,\xi} := E(\xi_t \otimes \xi_t \otimes \xi_t \otimes \xi_t)$ with n_ξ^2, n_ξ^3 and n_ξ^4 elements, respectively. In order to compute these objects efficiently, we first reduce the dimension of ξ_t, since it has some duplicate elements. That is, we compute product-moments for the $n_{\tilde{\xi}} = n_u + n_u(n_u+1)/2 + 2n_x n_u + n_x^2 n_u + n_x n_u^2 + n_u(n_u+1)(n_u+2)/6$ vector

$$\tilde{\xi}_{t+1} := \begin{pmatrix} u_{t+1} \\ DP_{n_u}^{+}(u_{t+1} \otimes u_{t+1} - \Gamma_{2u}) \\ \hat{x}_t^f \otimes u_{t+1} \\ \hat{x}_t^s \otimes u_{t+1} \\ \hat{x}_t^f \otimes \hat{x}_t^f \otimes u_{t+1} \\ \hat{x}_t^f \otimes u_{t+1} \otimes u_{t+1} \\ TP_{n_u}^{+}(u_{t+1} \otimes u_{t+1} \otimes u_{t+1} - \Gamma_{3,u}) \end{pmatrix}$$

given that

$$F_\xi = \begin{bmatrix} I_u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & DP_u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{xu} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{ux} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xu} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{ux} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & DP_x \otimes I_u & 0 & 0 \\ 0 & 0 & 0 & 0 & (I_x \otimes K_{ux})(DP_x \otimes I_u) & 0 & 0 \\ 0 & 0 & 0 & 0 & (K_{ux} \otimes I_x)(I_x \otimes K_{ux})(DP_x \otimes I_u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (I_x \otimes DP_u) & 0 \\ 0 & 0 & 0 & 0 & 0 & (K_{ux} \otimes I_u)(I_x \otimes DP_u) & 0 \\ 0 & 0 & 0 & 0 & 0 & (I_u \otimes K_{ux})(K_{ux} \otimes I_u)(I_x \otimes DP_u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & TP_u \end{bmatrix}$$

since

$$\xi_t = F_\xi \cdot \tilde{\xi}_t$$

and with $DP_{n_u}^{+}$ being the Moore-Penrose-Inverse of the duplication matrix DP_{n_u}, $TP_{n_u}^{+}$ being the Moore-Penrose-Inverse of the triplication matrix TP_{n_u} and K_{n_x,n_u} the commutation matrix such that $K_{n_x,n_u}(x_t^f \otimes u_{t+1}) = (u_{t+1} \otimes x_t^f)$. Then we have

$$M_{k,\xi} := [\otimes_{j=1}^{k} F_\xi] \cdot M_{k,\tilde{\xi}}$$

denoting the k-th (k=2,3,4)-order product moment of $\tilde{\xi}_t$. Since $[\otimes_{j=1}^{k} F_\xi]$ does not change with θ, we can focus on $M_{k,\tilde{\xi}}$. $M_{k,\tilde{\xi}}$, however, contains also many duplicate elements. Denote with $\widetilde{M}_{k,\tilde{\xi}}$ the unique elements of $M_{k,\tilde{\xi}}$, we have the following relationships:

$$M_{2,\tilde{\xi}} = DP_{n_{\tilde{\xi}}} \cdot \widetilde{M}_{2,\tilde{\xi}}, \quad M_{3,\tilde{\xi}} = TP_{n_{\tilde{\xi}}} \cdot \widetilde{M}_{3,\tilde{\xi}}, \quad M_{4,\tilde{\xi}} = QP_{n_{\tilde{\xi}}} \cdot \widetilde{M}_{4,\tilde{\xi}},$$

with the duplication matrix $DP_{n_{\tilde{\xi}}}$ defined by Magnus & Neudecker (1999), and the triplication matrix $TP_{n_{\tilde{\xi}}}$ and quadruplication matrix $QP_{n_{\tilde{\xi}}}$ similarly defined by Meijer (2005).[2] Note that these matrices are

[2] Actually $\widetilde{M}_{k,\tilde{\xi}}$ has some further duplicate terms for $n_u, n_x > 1$ due to higher-order cross terms of u_{t+1} and x_t^f, which we can further reduce using indices from the `unique` function of Matlab.

independent of θ and their Moore-Penrose-Inverse always exists, e.g. $(QP'_{n_{\tilde{\xi}}} QP_{n_{\tilde{\xi}}})^{-1} QP'_{n_{\tilde{\xi}}} \cdot M_{4,\tilde{\xi}} = \widetilde{M}_{4,\tilde{\xi}}$. Further, $DP_{n_{\tilde{\xi}}}$, $TP_{n_{\tilde{\xi}}}$ and $QP_{n_{\tilde{\xi}}}$ are constructed such that there is a unique ordering in $\widetilde{M}_{k,\tilde{\xi}}$, see Meijer (2005) for an example and more details.

The product-moments of $\tilde{\xi}_t$ can thus be computed symbolically as outlined in the second-order approximation.

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